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Mixed boundary-value problems of thermoelasticity for anisotropic-in-plan inhomogeneous toroidal shells☆

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ABSTRACT

Problems of thermoelasticity for an anisotropic-in-plan inhomogeneous thin toroidal shell are solved by asymptotic integration of the equations of the three-dimensional problem of the theory of an anisotropic inhomogeneous solid for various boundary conditions. Recurrence formulae are derived for the components of the asymmetric stress tensor and the displacement vector. An example is given.

Many publications[1] have been devoted to determining the stress–strain state of toroidal shells using classical shell theory. In this paper, we solve mixed boundary-value problems for a thin toroidal shell, taking into account the features of the geometry of a torus[2] (the sign variability of the Gaussian curvature). The method of asymptotic integration[3–5] of the equations of the three-dimensional problem of the theory of the thermoelasticity of an inhomogeneous anisotropic solid is used.

Consider a toroidal shell, the middle surface of which is obtained by rotating the generatrix of a circle of radius r around the OZ axis, lying in the plane of the circle at a distance R from its center. Suppose that, in a toroidal system of coordinates, it occupies the region

$$\Omega_* = \left\{\theta,\varphi,\gamma : |\theta| \le \pi, \quad 0 \le \varphi \le 2\pi, \ |\gamma| \le h, \quad h \ll r\right\}$$

In the figure we show a toroidal shell and its cross section, where the γ axis is directed along the outward normal to the middle surface of the shell, θ is the angle between the γ and OZ axes, and φ is the angle of rotation of the circle generatrix. The regions $0 < \theta < \pi$ and $-\pi < \theta < 0$ of the surface have positive and negative Gaussian curvature respectively.

The material of the shell is anisotropic and inhomogeneous in plan along the longitudinal θ, φ coordinates.

We will consider two cases:

1) on the outer surface $\gamma = h$ and the inner surface $\gamma = -h$ of the shell we are given the displacements

$$u_j(\theta,\varphi,\gamma = \pm h) = u_j^{\pm}, \quad j = \theta,\varphi,\gamma \tag{1}$$

2) on the outer surface we are given the displacements

$$u_j(\theta,\varphi,\gamma = h) = 0, \quad j = \theta,\varphi,\gamma \tag{2}$$

and a load

$$\sigma_{j\gamma}(\theta,\varphi,\gamma = -h) = \sigma_{j\gamma}^{-}(\theta,\varphi), \quad j = \theta,\varphi,\gamma \tag{3}$$

is applied to the inner surface (Fig. 1).

The surface of the shell is assumed to be continuous and closed, and hence no other conditions are imposed. A thermal field acts on the shell. The effect of this field is taken into account using the Duhamel-Neumann model, and it is assumed that the temperature function $\upsilon(\theta, \varphi, \gamma)$ is known and satisfies the heat-conduction equation and the corresponding boundary conditions.

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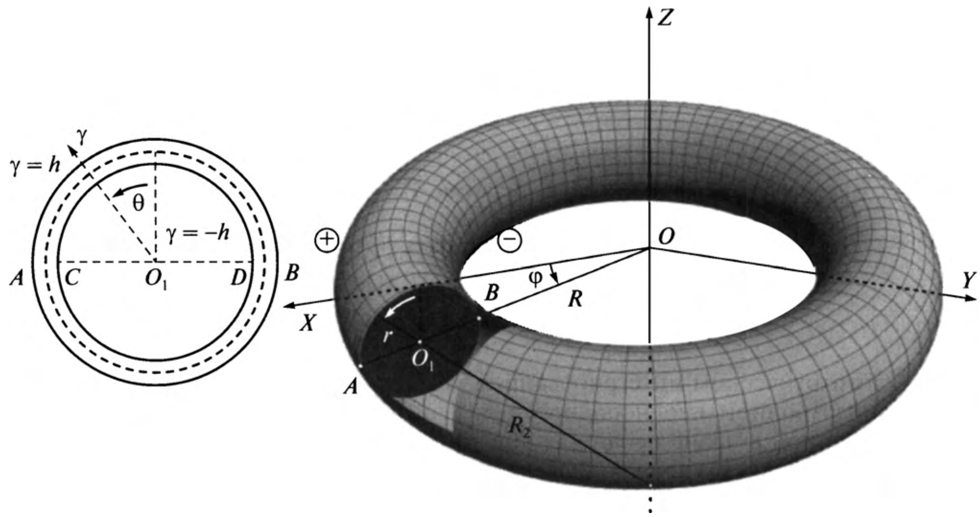

Fig. 1.

It is required to find the stress-strain state of the shell.

We will write the equations of equilibrium of an anisotropic body in curvilinear coordinates taking into account the bulk forces and the elasticity relation together with the temperature stresses

$$\frac{\partial}{\partial\alpha}(H_2H_3\sigma_{\alpha\alpha})+\frac{\partial}{\partial\beta}(H_1H_3\sigma_{\alpha\beta})+\frac{\partial}{\partial\gamma}(H_1H_2\sigma_{\alpha\gamma})-\sigma_{\beta\beta}H_3\frac{\partial H_2}{\partial\alpha}$$
$$-\sigma_{\gamma\gamma}H_2\frac{\partial H_3}{\partial\alpha}+\sigma_{\alpha\beta}H_3\frac{\partial H_1}{\partial\beta}+\sigma_{\alpha\gamma}H_2\frac{\partial H_1}{\partial\gamma}+P_\alpha H_1H_2H_3=0 \quad (\alpha,\beta,\gamma;1,2,3)$$
$$H_1=A\left(1+\frac{\gamma}{R_1}\right),\quad H_2=B\left(1+\frac{\gamma}{R_2}\right),\quad H_3=1$$
$$\mathrm{col}[e_{\alpha\alpha}-\alpha_{11}\vartheta,\ldots,e_{\alpha\beta}-\alpha_{12}\vartheta]=\|a_{ij}\|_{6\times6}\mathrm{col}[\sigma_{\alpha\alpha},\sigma_{\beta\beta},\ldots,\sigma_{\alpha\beta}]$$
$$e_{\alpha\alpha}=\frac{1}{H_1}\frac{\partial u_\alpha}{\partial\alpha}+\frac{1}{H_1H_2}\frac{\partial H_1}{\partial\beta}u_\beta+\frac{1}{H_1H_3}\frac{\partial H_1}{\partial\gamma}u_\gamma,\quad e_{\alpha\beta}=\frac{H_1}{H_2}\frac{\partial}{\partial\beta}\left(\frac{u_\alpha}{H_1}\right)+\frac{H_2}{H_1}\frac{\partial}{\partial\alpha}\left(\frac{u_\beta}{H_2}\right)(\alpha,\beta,\gamma;1,2,3) \tag{4}$$

Here H_1, H_2 and H_3 are Lamé coefficients, A and B are the coefficients of the first quadratic form of the coordinate surface, R_1 and R_2 are the principal radii of curvature of the coordinate surface, a_{ij} are the elastic pliability coefficients, α_{ij} ($i, j = 1, 2, 3$) are the linear thermal expansion coefficients and P_α, P_β, P_γ are the components of the bulk force.

For the toroidal shell considered, $\alpha = \theta$ and $\beta = \varphi$, and also

$$A=r,\quad B=R+r\sin\theta,\quad R_1=r,\quad R_2=(R+r\sin\theta)/\sin\theta$$
$$a_{ij}=a_{ij}(\theta,\varphi),\quad i,j=1,2,\ldots,6;\quad \alpha_{ij}=\alpha_{ij}(\theta,\varphi)\quad i,j=1,2,3 \tag{5}$$

We will replace the components of the symmetrical stress tensor $\sigma_{\theta\gamma} = \sigma_{\gamma\theta}$ (θ, φ, γ) by the components of the asymmetrical tensor $\tau_{\theta\gamma} \neq \tau_{\gamma\theta}$ using the formulae[2,3]

$$\tau_{\theta j}=(1+\gamma/R_2)\sigma_{\theta j},\quad \tau_{\varphi j}=(1+\gamma/R_1)\sigma_{\varphi j},\quad j=\theta,\varphi,\gamma,\quad \tau_{\gamma\gamma}=(1+\gamma/R_1)(1+\gamma/R_2)\sigma_{\gamma\gamma} \tag{6}$$

and we will change to dimensionless coordinates and dimensionless strains using the formulae

$$\xi=\theta,\quad \eta=\varphi,\quad \zeta=\frac{\gamma}{h}=\varepsilon^{-1}\frac{\gamma}{r},\quad u=\frac{u_\theta}{r},\quad \upsilon=\frac{u_\varphi}{r},\quad w=\frac{u_\gamma}{r},\quad \varepsilon=\frac{h}{r};\quad h\ll r \tag{7}$$

after which the equations and relations (4) for the torus take the form

$$\begin{aligned}
&\frac{\partial\tau_{\theta\theta}}{\partial\theta}+\frac{r\cos\theta}{B}(\tau_{\theta\theta}-\tau_{\varphi\varphi})+\frac{r}{B}\frac{\partial\tau_{\varphi\theta}}{\partial\varphi}+2\tau_{\theta\gamma}+\varepsilon^{-1}\Lambda_1\frac{\partial\tau_{\theta\gamma}}{\partial\zeta}+rP_\theta=0\\
&\frac{\partial\tau_{\theta\varphi}}{\partial\theta}+\frac{r\cos\theta}{B}(\tau_{\theta\varphi}+\tau_{\varphi\theta})+\frac{r}{B}\frac{\partial\tau_{\varphi\varphi}}{\partial\varphi}+2\tau_{\varphi\gamma}\frac{r\sin\theta}{B}+\varepsilon^{-1}\Lambda_2\frac{\partial\tau_{\theta\gamma}}{\partial\zeta}+rP_\varphi=0\\
&\frac{\partial\tau_{\theta\gamma}}{\partial\theta}-\left(\tau_{\theta\theta}+\tau_{\varphi\varphi}\frac{r\sin\theta}{B}\right)+\frac{r}{B}\frac{\partial\tau_{\varphi\gamma}}{\partial\varphi}+\tau_{\theta\gamma}\frac{r\cos\theta}{B}+\varepsilon^{-1}\frac{\partial\tau_{\gamma\gamma}}{\partial\zeta}+rP_\gamma=0\\
&\Lambda_2\left(\frac{\partial u}{\partial\theta}+w\right)=e_1+\alpha_{11}\Lambda\vartheta+\varepsilon\zeta U_1\\
&\frac{\Lambda_1}{B}\left(\frac{\partial\upsilon}{\partial\varphi}+u\cos\theta+w\sin\theta\right)=e_2+\alpha_{22}\Lambda\vartheta+\varepsilon\zeta U_2\\
&\frac{\Lambda_1}{B}\left(\frac{\partial u}{\partial\varphi}-\upsilon\cos\theta\right)+\Lambda_2\frac{\partial\upsilon}{\partial\theta}=e_6+\alpha_{12}\Lambda\vartheta+\varepsilon\zeta U_6\\
&\varepsilon^{-1}\Lambda\frac{\partial w}{\partial\zeta}=e_3+\alpha_{33}\Lambda\vartheta+\varepsilon\zeta U_3\\
&\varepsilon^{-1}\Lambda\frac{\partial\upsilon}{\partial\zeta}-\Lambda_1\frac{r}{B}\left(\upsilon\sin\theta-\frac{\partial w}{\partial\varphi}\right)=e_4+\alpha_{23}\Lambda\vartheta+\varepsilon\zeta U_4\\
&\varepsilon^{-1}\Lambda\frac{\partial u}{\partial\zeta}-\Lambda_2\left(u-\frac{\partial w}{\partial\theta}\right)=e_5+\alpha_{13}\Lambda\vartheta+\varepsilon\zeta U_5\\
&e_j=a_{j1}\tau_{\theta\theta}+a_{j2}\tau_{\varphi\varphi}+a_{j3}\tau_{\gamma\gamma}+a_{j4}\tau_{\varphi\gamma}+a_{j5}\tau_{\theta\gamma}+a_{j6}\tau_{\theta\varphi}\\
&U_j=a_{1j}\tau_{\theta\theta}+\frac{r\sin\theta}{B}\left(a_{2j}\tau_{\varphi\varphi}+a_{j4}\tau_{\varphi\gamma}\right)+a_{j5}\tau_{\theta\gamma}+a_{j6}\tau_{\theta\varphi};\quad j=3,4,5,6\\
&\Lambda_1=1+\varepsilon\zeta,\quad \Lambda_2=1+\varepsilon\zeta\frac{r\sin\theta}{B},\quad \Lambda=1+\varepsilon\zeta\left(1+\frac{r\sin\theta}{B}\right)+\varepsilon^2\zeta^2\frac{r\sin\theta}{B}
\end{aligned} \tag{8}$$

The system of equations and relations (8) are singularly perturbed by the geometrical small parameter ε. Consequently,[2–7] the solution is made up of two solutions: an internal solution, predominant inside the region occupied by the thin body, and the solution of the problem for the boundary layer, which decreases exponentially in the direction of the inward normal to the end surfaces.[6,7] The surface of the toroidal shell is continuous and closed, and hence here we will only construct the internal solution, which we will seek in the form of the asymptotic expansion[4]

$$Q=\sum_{s=0}^{S}\varepsilon^{\chi_Q+s}Q^{(s)}(\theta,\varphi,\zeta),\quad \chi_u=0,\ \chi_\sigma=-1 \tag{9}$$

where Q is any of the stresses and dimensionless strains.

The bulk forces and the temperature function can simultaneously be represented in the form

$$P_\theta=\sum_{s=0}^{S}\varepsilon^{-2+s}r^{-1}P_{\theta s}(\theta,\varphi,\zeta)(\theta,\varphi,\gamma),\quad \vartheta=\sum_{s=0}^{S}\varepsilon^{-1+s}\vartheta^{(s)}(\theta,\varphi,\zeta) \tag{10}$$

It follows from relations (10) that the order of the stress-strain state, due to the volume forces and the temperature field, will be of the order of the stress-strain state produced by the strains and forces applied to the faces, if the intensity of the bulk forces is a factor of ε^{-2} greater than the order of the dimensionless boundary strains, while the intensity of the change in the temperature field is a factor of ε^{-1} greater than the order of the dimensionless boundary strains. Otherwise, the contribution of the first will be less and the corresponding terms will occur in the equations for higher approximations.

Substituting expressions (9) and (10) into relations (8) and equating coefficients of like powers of ε on both sides of each equation, we obtain a non-contradictory system of equations in the unknown coefficients $Q^{(s)}$ of expansion (9). Solving this system, we obtain the following recurrence formulae for determining $Q^{(s)}$, i.e., the components of the stress tensor and the strain vector are as follows:

$$\begin{aligned}
&\tau_{\theta\theta}^{(s)} = \tilde{\tau}_1^{(s)} + \tau_{\theta\theta*}^{(s)}(\theta,\varphi,\zeta) \quad (\theta\theta,\varphi\varphi,\theta\varphi;1,2,6)\\
&\tau_{\gamma\gamma}^{(s)} = \tau_{\gamma\gamma 0}^{(s)}(\theta,\varphi) + \tau_{\gamma\gamma*}^{(s)}(\theta,\varphi,\zeta)\\
&\tau_{\theta\gamma}^{(s)} = \tau_{\theta\gamma 0}^{(s)}(\theta,\varphi) + \tau_{\theta\gamma*}^{(s)}(\theta,\varphi,\zeta) \quad (\theta,\varphi)\\
&u^{(s)} = \zeta\tilde{\tau}_5 + u_0^{(s)}(\theta,\varphi) + u_*^{(s)}(\theta,\varphi,\zeta) \quad (u,\upsilon,w;5,4,3)\\
&\tilde{\tau}_j^{(s)} = A_{j3}\tau_{\gamma\gamma 0}^{(s)} + A_{j4}\tau_{\varphi\gamma 0}^{(s)} + A_{j5}\tau_{\theta\gamma 0}^{(s)}, \quad j = 1,2,\ldots,6
\end{aligned} \tag{11}$$

Here

$$\begin{aligned}
&\tau_{\theta\theta*}^{(s)} = B_{11}P_1^{(s)} + B_{12}P_2^{(s)} + B_{16}P_3^{(s)}, \quad \tau_{\varphi\varphi*}^{(s)} = B_{12}P_1^{(s)} + B_{22}P_2^{(s)} + B_{26}P_3^{(s)},\\
&\tau_{\theta\varphi*}^{(s)} = B_{16}P_1^{(s)} + B_{26}P_2^{(s)} + B_{66}P_3^{(s)}\\
&\tau_{\gamma\gamma*}^{(s)} = -\int_0^\zeta\left[\frac{\partial\tau_{\varphi\gamma}^{(s-1)}}{\partial\theta} - \left(\tau_{\theta\theta}^{(s-1)} + \tau_{\varphi\varphi}^{(s-1)}\frac{r\sin\theta}{B}\right) + \frac{r}{B}\frac{\partial\tau_{\varphi\gamma}^{(s-1)}}{\partial\varphi} + \tau_{\theta\gamma}^{(s-1)}\frac{r\cos\theta}{B} + rP_\gamma^{(s)} + rL\left(P_\gamma^{(s-1)}\right)\right]d\zeta\\
&\tau_{\theta\gamma*}^{(s)} = -\int_0^\zeta\left[\frac{\partial\tau_{\theta\theta}^{(s-1)}}{\partial\theta} + \frac{r\cos\theta}{B}\left(\tau_{\theta\theta}^{(s-1)} - \tau_{\varphi\varphi}^{(s-1)}\right) + \frac{r}{B}\frac{\partial\tau_{\varphi\theta}^{(s-1)}}{\partial\varphi} + 2\tau_{\theta\gamma}^{(s-1)} + \zeta\frac{\partial\tau_{\theta\gamma}^{(s-1)}}{\partial\zeta} + rP_\theta^{(s)} + rL\left(P_\theta^{(s-1)}\right)\right]d\zeta\\
&\tau_{\varphi\gamma*}^{(s)} = -\int_0^\zeta\left[\frac{\partial\tau_{\theta\varphi}^{(s-1)}}{\partial\theta} + \frac{r\cos\theta}{B}\left(\tau_{\theta\varphi}^{(s-1)} + \tau_{\varphi\theta}^{(s-1)}\right) + \frac{r}{B}\frac{\partial\tau_{\varphi\varphi}^{(s-1)}}{\partial\varphi}\right.\\
&\left.+\frac{r\sin\theta}{B}\left(2\tau_{\varphi\gamma}^{(s-1)} + \zeta\frac{\partial\tau_{\theta\gamma}^{(s-1)}}{\partial\zeta}\right) + rP_\varphi^{(s)} + rL\left(P_\varphi^{(s-1)}\right)\right]d\zeta\\
&P_1^{(s)} = L_2\left(\frac{\partial u^{(s-1)}}{\partial\theta} + w^{(s-1)}\right) - a_{13}\tau_{\gamma\gamma*}^{(s)} - a_{14}\tau_{\varphi\gamma*}^{(s)} - a_{15}\tau_{\theta\gamma}^{(s)} - \alpha_{11}\left(\vartheta^{(s)} - L\left(\vartheta^{(s-1)}\right)\right) - \zeta U_1^{(s-1)}\\
&P_2^{(s)} = \frac{1}{B}L_1\left(\frac{\partial\upsilon^{(s-1)}}{\partial\varphi} + u^{(s-1)}\cos\theta + w^{(s-1)}\sin\theta\right) - a_{23}\tau_{\gamma\gamma*}^{(s)} - a_{24}\tau_{\varphi\gamma*}^{(s)} - a_{25}\tau_{\theta\gamma*}^{(s)}\\
&-\alpha_{22}\left(\vartheta^{(s)} - L\left(\vartheta^{(s-1)}\right)\right) - \zeta U_2^{(s-1)}\\
&P_3^{(s)} = \frac{1}{B}L_1\left(\frac{\partial u^{(s-1)}}{\partial\varphi} - \upsilon^{(s-1)}\cos\theta\right) + L_2\left(\frac{\partial\upsilon^{(s-1)}}{\partial\theta}\right) - a_{36}\tau_{\gamma\gamma*}^{(s)} - a_{46}\tau_{\varphi\gamma*}^{(s)} - a_{56}\tau_{\theta\gamma*}^{(s)}\\
&-\alpha_{12}\left(\vartheta^{(s)} - L\left(\vartheta^{(s-1)}\right)\right) - \zeta U_6^{(s-1)}\\
&u_*^{(s)} = \int_0^\zeta\left[\Lambda_{u51}^{(s)} + L_2\left(u^{(s-1)} - \frac{\partial w^{(s-1)}}{\partial\theta}\right)\right]d\zeta, \quad \upsilon_*^{(s)} = \int_0^\zeta\left[\Lambda_{\upsilon 42}^{(s)}\frac{r}{B}L_1\left(\upsilon^{(s-1)}\sin\theta - \frac{\partial w^{(s-1)}}{\partial\varphi}\right)\right]d\zeta\\
&w_*^{(s)} = \int_0^\zeta\Lambda_{w33}^{(s)}d\zeta\\
&\Lambda_{u\ln}^{(s)} = e_{l*}^{(s)} + \alpha_{n3}\vartheta^{(s)} + L\left(\alpha_{n3}\vartheta^{(s-1)} - \frac{\partial u^{(s-1)}}{\partial\zeta}\right) + \zeta U_l^{(s-1)}(u,\upsilon,w), \quad l = 3,4,5, \quad n = 1,2,3\\
&U_j^{(s-1)} = a_{1j}\tau_{\theta\theta}^{(s-1)} + \frac{r\sin\theta}{B}\left(a_{2j}\tau_{\varphi\varphi}^{(s-1)} + a_{j4}\tau_{\varphi\gamma}^{(s-1)}\right) + a_{j5}\tau_{\theta\gamma}^{(s-1)} + a_{j6}\tau_{\theta\varphi}^{(s-1)}, \; j = 3,4,5,6\\
&B_{pj} = \left(a_{pk}a_{jk} - a_{pj}a_{kk}\right)/\Delta \quad (p \neq j \neq k \neq p)\\
&B_{kk} = \left(a_{pp}a_{jj} - a_{pj}^2\right)/\Delta, \quad B_{pj} = B_{jp}; \quad p,j,k = 1,2,6\\
&A_{kl} = -a_{1l}B_{k1} - a_{2l}B_{k2} - a_{6l}B_{k6},\\
&A_{ml} = a_{ml}A_{1l} + a_{m2}A_{2l} + a_{m6}A_{6l} + a_{ml}, \quad A_{ml} \neq A_{lm}; \quad l,m = 3,4,5\\
&\Delta = a_{11}a_{22}a_{66} + 2a_{12}a_{16}a_{26} - a_{22}a_{16}^2 - a_{66}a_{12}^2 - a_{11}a_{26}^2
\end{aligned} \tag{12}$$

We will denote the following operators by L_1, L_2 and L:

$$L_1\left(Q^{(s-1)}\right)=Q^{(s-1)}+\zeta Q^{(s-2)},\quad L_2\left(Q^{(s-1)}\right)=Q^{(s-1)}+\zeta\frac{r\sin\theta}{B}Q^{(s-2)}$$
$$L\left(Q^{(s-1)}\right)=\zeta\left(1+\frac{r\sin\theta}{B}\right)Q^{(s-1)}+\zeta^2\frac{r\sin\theta}{B}Q^{(s-2)} \tag{13}$$

The common integral (11) contains 6S integration functions

$$\tau_{j\gamma 0}^{(s)},\quad j=\theta,\varphi,\gamma,\quad u_0^{(s)},\upsilon_0^{(s)},w_0^{(s)} \tag{14}$$

which are uniquely defined from the conditions specified on the shell surfaces.

Satisfying the kinematic boundary conditions (1), we obtain the values of the integration functions in the form

$$\tau_{\gamma\gamma 0}^{(s)}=W_3^{(s)}/\Delta_*,\quad \tau_{\varphi\gamma 0}^{(s)}=W_4^{(s)}/\Delta_*,\quad \tau_{\theta\gamma 0}^{(s)}=W_5^{(s)}/\Delta_*,\quad \Delta_*=A_{33}B_{33}^*+A_{34}B_{43}^*+A_{35}B_{53}^*$$
$$W_j^{(s)}=B_{j5}^*V_\theta^{(s)}+B_{j4}^*V_\varphi^{(s)}+B_{j3}^*V_\gamma^{(s)},\quad j=3,4,5$$
$$u_0^{(s)}=u^{+(s)}-\hat{U}_5^{(s)}-u_*^{(s)}(\zeta=1)\quad (u,\upsilon,w;5,4,3)$$
$$B_{jk}^*=A_{jk}A_{ll}-A_{jl}A_{kl},\quad B_{ll}^*=A_{jk}A_{kj}-A_{jj}A_{kk},\quad B_{kj}^*\neq B_{jk}^*$$
$$j,k,l=3,4,5\quad (l\neq j\neq k\neq l)$$
$$V_\theta^{(s)}=\frac{1}{2}\left(u^{+(s)}-u^{-(s)}+u_*^{(s)}(\zeta=-1)-u_*^{(s)}(\zeta=1)\right)\quad (\theta,\varphi,\gamma;u,\upsilon,w)$$
$$u^{\pm(s)}=u^{\pm}/r,\quad u^{\pm(s)}=0,\quad s>0\quad (u,\upsilon,w)$$
$$\hat{U}_j^{(s)}=A_{j3}\tau_{\gamma\gamma 0}^{(s)}+A_{j4}\tau_{\varphi\gamma 0}^{(s)}+A_{j5}\tau_{\theta\gamma 0}^{(s)},\quad j=5,4,3 \tag{15}$$

which will have the following form for the mixed boundary conditions (2) and (3)

$$\tau_{j\gamma 0}^{(s)}=\sigma_{j\gamma}^{-(s)}-\tau_{j\gamma *}^{(s)}(\zeta=-1),\quad j=\theta,\varphi,\gamma$$
$$u_0^{(s)}=u^{+(s)}-\hat{U}_5^{(s)}-u_*^{(s)}(\zeta=1)\quad (u,\upsilon,w;5,4,3)$$
$$\sigma_{\theta\gamma}^{-(0)}=\varepsilon\bar{\sigma}_{\theta\gamma},\quad j=\theta,\varphi,\gamma,\quad \sigma_{\gamma\gamma}^{-(1)}=-(1+\Phi)\varepsilon\bar{\sigma}_{\gamma\gamma},\quad \sigma_{\gamma\gamma}^{-(2)}=\Phi\varepsilon\bar{\sigma}_{\gamma\gamma},\quad \sigma_{\gamma\gamma}^{-(s)}=0,\quad s>2$$
$$\sigma_{\theta\gamma}^{-(1)}=-\Phi\varepsilon\bar{\sigma}_{\theta\gamma},\quad \sigma_{\varphi\gamma}^{-(1)}=-\varepsilon\bar{\sigma}_{\varphi\gamma},\quad \sigma_{j\gamma}^{-(s)}=0,\quad s>1,\quad j=\theta,\varphi;\quad \Phi=\frac{r\sin\theta}{B} \tag{16}$$

We will give some examples of the stress-strain state of an orthotropic toroidal shell for two iteration steps.

Example 1. A normal pressure acts on the inner surface of an orthotropic shell, while the outer surface is rigidly clamped. We have

$$\gamma=-h:\sigma_{\gamma\gamma}=\sigma=-p,\ \sigma_{\theta\gamma}=\sigma_{\varphi\gamma}=0;\ \gamma=h:\ u_j=0,\ j=\theta,\varphi,\gamma \tag{17}$$

After the first iteration step we obtain

$$\tau_{\theta\theta}^{(0)}=A_{13}\varepsilon\sigma,\quad \tau_{\varphi\varphi}^{(0)}=A_{23}\varepsilon\sigma,\quad \tau_{\gamma\gamma}^{(0)}=\varepsilon\sigma,\quad \sigma=-p$$
$$\tau_{\theta\varphi}^{(0)}=\tau_{\theta\gamma}^{(0)}=\tau_{\varphi\gamma}^{(0)}=0,\quad u^{(0)}=\upsilon^{(0)}=0,\quad w^{(0)}=A_{33}(\zeta-1)\varepsilon\sigma \tag{18}$$

After the second step

$$\tau_{\theta\theta}^{(1)}=X_1,\ \tau_{\varphi\varphi}^{(1)}=X_2,\quad \tau_{\gamma\gamma}^{(1)}=\varepsilon\sigma(\zeta+1)\left(A_{13}+A_{23}\frac{r\sin\theta}{B}\right)-\varepsilon\sigma\left(1+\frac{r\sin\theta}{B}\right)$$
$$\tau_{\theta\varphi}^{(1)}=0,\quad \tau_{\theta\gamma}^{(1)}=\varepsilon(\zeta+1)\left[\sigma(A_{13}+A_{23})\frac{r\cos\theta}{B}-A_{13}\frac{d\sigma}{d\theta}\right],\quad \tau_{\varphi\gamma}^{(1)}=0$$
$$u^{(1)}=a_{55}\varepsilon\frac{(\zeta-1)(\zeta+3)}{2}\left[\sigma(A_{23}-A_{13})\frac{r\cos\theta}{B}-A_{13}\frac{d\sigma}{d\theta}\right]-\varepsilon A_{33}\frac{(\zeta-1)^2}{2}\frac{d\sigma}{d\theta},\quad \upsilon^{(1)}=0 \tag{19}$$

Table 1

μ	$h/r = 0.1$		$h/r = 0.01$	
	$R/r = 2$	3/2	2	3/2
μ_A	0.968	0.962	0.997	0.996
μ_B	1.111	1.250	1.010	1.020
μ_C	1.034	1.042	1.003	1.004
μ_D	0.909	0.833	0.990	0.980

$$w^{(1)} = A_{33}\varepsilon\sigma(\zeta-1)\left[\left(1+\frac{r\sin\theta}{B}\right)+2\left(A_{13}+A_{23}\frac{r\sin\theta}{B}\right)\right]+\varepsilon\sigma\frac{\zeta^2-1}{2}\left(A_{13}^*+A_{23}^*\frac{r\sin\theta}{B}\right)$$

$$X_n = A_{n3}\tau_{\gamma\gamma}^{(1)}(\theta,\zeta=0)+A_{33}\varepsilon\sigma(\zeta-1)\left(B_{1n}+B_{n2}\frac{r\sin\theta}{B}\right)$$
$$+\zeta\varepsilon\sigma\left[A_{13}(A_{n1}+A_{n3})+A_{23}(A_{n2}+A_{23})\frac{r\sin\theta}{B}\right],\quad n=1,2$$

$$A_{kl} = -a_{il}B_{kl}-a_{2l}B_{k2},\quad A_{kl}\neq A_{lk},\quad k,l=1,2$$

$$A_{m3}^* = a_{m1}A_{13}+a_{m2}A_{23}+a_{m3},\quad A_{mj}^*\neq A_{jm}^*\quad m,j=1,2,3$$

Hence, after two iteration steps the components of the asymnmetrical stress tensor and the displacement vector will be

$$\tau_{\theta\theta}=\frac{r}{h}\tau_{\theta\theta}^{(0)}+\tau_{\theta\theta}^{(1)},\quad \tau_{\varphi\varphi}=\frac{r}{h}\tau_{\varphi\varphi}^{(0)}+\tau_{\varphi\varphi}^{(1)},\quad \tau_{\gamma\gamma}=\frac{r}{h}\tau_{\gamma\gamma}^{(0)}+\tau_{\gamma\gamma}^{(1)},\quad \tau_{\theta\varphi}=\tau_{\theta\varphi}^{(1)}$$
$$\tau_{\theta\gamma}=\tau_{\theta\gamma}^{(1)},\quad \tau_{\varphi\gamma}=\tau_{\varphi\gamma}^{(1)},\quad u_\theta=hu^{(1)},\quad u_\varphi=h\upsilon^{(1)},\quad u_\gamma=rw^{(0)}+hw^{(1)} \tag{20}$$

Example 2. A hydrostatic pressure acts on the inner surface of the shell, while the outer surface is rigidly clamped. In this case the components of the stress tensor and the displacement vector are given by formulae (18)–(20), where we must take

$$\sigma = -\rho gr(1-\cos\theta) \tag{21}$$

Example 3. The inner surface of the shell is heated uniformly to a temperature $\upsilon(\theta,\varphi,\gamma)$ and rigidly clamped, while the outer surface is load-free. We have

$$u_j(\theta,\varphi,\gamma=-h)=0,\quad \sigma_{j\gamma}(\theta,\varphi,\gamma=h)=0,\quad j=\theta,\varphi,\gamma \tag{22}$$

After the first iteration step the components of the stress tensor and the displacement vector will be

$$\tau_{\theta\theta}=-C_1\vartheta,\quad \tau_{\varphi\varphi}=-C_2\vartheta,\quad \tau_{\gamma\gamma}=\tau_{\varphi\gamma}=\tau_{\theta\gamma}=0,\quad \tau_{\theta\varphi}=-C_6\vartheta$$
$$\frac{u_\theta}{\alpha_{13}-e_4}=\frac{u_\varphi}{\alpha_{23}-e_5}=\frac{u_\gamma}{\alpha_{33}-e_3}=\int_{-h}^{\gamma}\vartheta d\gamma$$
$$C_j=B_{j1}\alpha_{11}+B_{j2}\alpha_{22}+B_{j6}\alpha_{12},\quad j=1,2,6$$
$$e_k=a_{k1}C_1+a_{k2}C_2+a_{k6}C_6,\quad k=3,4,6 \tag{23}$$

If the outer surface of the shell is rigidly clamped, while the inner surface is load-free, the components of the stress and displacement fields will be given by formulae (23), where $-h$ must be replaced by h.

Hence, theoretical recurrence formulae (9)–(13) enable us to calculate the components of the displacement vector and the stress tensor of a toroidal shell with any asymptotic accuracy specified in advance, when the conditions of the first or second boundary-value problems of the theory of thermoelasticity are specified on its surface. If combinations of the mixed boundary conditions, consisting of conditions (1), (2) and (3), are specified on its outer and inner surfaces, the components of the stress and displacement fields will be given by recurrence formulae (15) or (16), depending on whether the components of the displacement vector or the stress tensor are specified (Table 1).

Note than when $R\to\infty$, the toroidal shell converts into a cylindrical shell of infinite length, for which all the problems formulated and their solutions remain true in the limit as $R\to\infty$.

An analysis of the stress-strain state of the toroidal shell shows that, in the region where the Gaussian curvature of the surface is negative ($-\pi<\theta<0$), the values of the stresses $\sigma_{\alpha\alpha}, \sigma_{\gamma\gamma}, \sigma_{\alpha\beta}, \sigma_{\alpha\gamma}$ are somewhat greater, while in the region of positive Gaussian curvature ($0<\theta<\pi$) they are less than their values at the corresponding points of a cylindrical shell. This can be seen from the table, where we show values of the ratios μ of the moduli of these stresses at corresponding points for a toroidal shell and for a cylindrical shell of radius r.

Recurrence formulae (9)–(16) are readily available algorithms for a computer program, which enables analytic and numerical solutions of the above boundary-value problems to be obtained fairly rapidly with any asymptotic accuracy $O(\varepsilon^S)$.

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